J Glob Optim (2007) 37:245–261
DOI 10.1007/s10898-006-9047-7

ORIGINAL PAPER

Some remarks on nonsmooth critical point theory

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Received: 31 January 2006 / Accepted: 15 May 2006 /
Published online: 5 August 2006

Abstract A general min–max principle established by Ghoussoub is extended to the case of functionals f which are the sum of a locally Lipschitz continuous term and of a convex, proper, lower semicontinuous function, when f satisfies a compactness condition weaker than the Palais–Smale one, i.e., the so-called Cerami condition. Moreover, an application to a class of elliptic variational–hemivariational inequalities in the resonant case is presented.

Keywords Critical points for nonsmooth functions · Nonsmooth Cerami condition · Elliptic variational–hemivariational inequalities · Problem at risonance

Mathematics Subject Classification (2000) 58E05 · 49J35

1 Introduction

Starting from the well-known mountain pass theorem (briefly, MPT) of Ambrosetti and Rabinowitz [1], many authors were interested in finding critical points of real-valued functions f defined on an infinite dimensional Banach space X, obtaining several generalizations of the MPT, which allow to solve wide classes of ordinary or partial differential equations, as well as variational or variational–hemivariational inequalities and elliptic equations with discontinuous nonlinearities. In particular, the existence of critical points was prevalently investigated along the following directions:

(a) the boundary conditions are relaxed (namely, certain inequalities in the min-max principles are allowed to be non-strict);

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(b) f is required to be nonsmooth;
(c) the ‘Palais–Smale condition’ (briefly, PS) is weakened.

When f is a C^1 function on X, the case (a) was completely studied by Pucci and Serrin [19, Theorem 1], Rabinowitz [20, Theorem 2.13], Ghoussoub and Preiss [11, Theorem 1.bis], Du [8, Theorem 2.1] and, for very general classes of *dual sets*, Ghoussoub [10, Theorem 1.bis].

In order to (b), a critical point theory for functions which are locally Lipschitz continuous on X was first developed by Chang [5]. Subsequently, Szulkin treated the case when f: $X \to \mathbb{R} \cup \{+\infty\}$ is a function having the following structure

$$f(u) := \Phi(u) + \psi(u) \quad \text{for all } u \in X, \tag{1}$$

where $\Phi \in C^1(X, \mathbb{R})$ and ψ: $X \to \mathbb{R} \cup \{+\infty\}$ is convex, proper, and lower semicontinuous (see [22]). Both [5, 22] were finally unified by Motreanu and Panagiotopoulos, which studied the more general case when f is like in (1), but Φ is only locally Lipschitz continuous on X (see [16]).

Inside the nonsmooth critical point theory, the study of (a) was successfully developed for locally Lipschitz continuous functions by Motreanu and Varga [18, Theorem 2.1] and Barletta and Marano [2, Theorem 4.1]. This investigation was continued in the Motreanu and Panagiotopoulos’ framework by Marano and Motreanu [14, Theorem 3.1], while in [13] the authors extend the general min-max principle of Ghoussoub [10] when, in addition, ψ is required to be continuous on every compact subset of X on which it is bounded.

Finally, first Cerami [4] and subsequently Bartolo et al. [3] in a similar way, introduced, for C^1 functions, a compactness condition, namely the ‘Cerami condition’ (briefly, (C)), which generalizes the usual (PS). In [12], Kourogenis and Papageorgiou extended the theory of Chang in direction (a) when the (PS) is replaced by (C). The (C) condition has been employed fruitfully, for example, by Schechter when f is C^1 (see [21]) and Marano and Papageorgiou when f is locally Lipschitz continuous (see [15]). However, to the best of our knowledge, nothing was said for the Motreanu and Panagiotopoulos’ setting when condition (C) is required instead of (PS).

The main purpose of this paper is to fill in such a gap. In Sect. 2, we introduce the (C) condition for functions having the same structure of those previously introduced in [13] and we state a deformation lemma (Theorem 2.1 below) which represents a useful version of Theorem 2.2 of [13]. In Sect. 3, we obtain a general critical point result (see Theorem 3.2) when (a) holds and (C) is assumed. An application to an elliptic variational–hemivariational inequality in the resonant case patterned after Problem (0.4) in [3] is then presented in Sect. 4.

2 Some preliminaries

Let $(X, \|\cdot\|)$ be a real Banach space. If V is a subset of X, we write $\mathrm{int}(V)$ for the interior of V, $\overline{V}$ for the closure of V, ∂V for the boundary of V. When V is nonempty, $x \in X$, and $\delta > 0$, we define $B(x, \delta) := \{z \in X : \|z - x\| < \delta\}$ as well as $B_\delta := B(0, \delta)$ and

$$d(x, V) := \inf_{z \in V} \|x - z\|, \qquad N_\delta(V) = \{z \in X : d(z, V) < \delta)\}.$$

Given $x, z \in X$, the symbol $[x, z]$ indicates the line segment joining x to z, namely

$$[x, z] := \{(1 - t)x + tz : t \in [0, 1]\}.$$

Moreover, $]x, z] := [x, z] \setminus \{x\}$. We denote by X^* the dual space of X, while $\langle \cdot, \cdot \rangle$ stands for the duality pairing between X^* and X. A function $\Phi\colon X \to \mathbb{R}$ is called locally Lipschitz continuous when to every $x \in X$ there correspond a neighbourhood V_x of x and a constant $L_x \geq 0$ such that

$$|\Phi(z) - \Phi(w)| \leq L_x \|z - w\| \quad \forall z, w \in V_x .$$

If $x, z \in X$, we write $\Phi^0(x; z)$ for the generalized directional derivative of Φ at the point x along the direction z, i.e.,

$$\Phi^0(x; z) := \limsup_{w \to x,\, t \to 0^+} \frac{\Phi(w + tz) - \Phi(w)}{t} .$$

It is known [6, Proposition 2.1.1] that Φ^0 is upper semicontinuous on $X \times X$. The generalized gradient of the function Φ in x, denoted by $\partial \Phi(x)$, is the set

$$\partial \Phi(x) := \left\{ x^* \in X^* : \langle x^*, z \rangle \leq \Phi^0(x; z) \ \forall z \in X \right\} .$$

Proposition 2.1.2 of [6] ensures that $\partial \Phi(x)$ turns out nonempty, convex, in addition to weak* compact. Hence, we can consider

$$m(x) = \inf\{\|x^*\| : x^* \in \partial \Phi(x)\}, \text{ being attained.}$$

Let f be a function on X satisfying the structural hypothesis

(H_f) $f(x) := \Phi(x) + \psi(x)$ for all $x \in X$, where $\Phi\colon X \to \mathbb{R}$ is locally Lipschitz continuous while $\psi\colon X \to \mathbb{R} \cup \{+\infty\}$ is convex, proper, and lower semicontinuous.

Put $D_\psi := \{x \in X : \psi(x) < +\infty\}$. Since ψ turns out continuous on $\mathrm{int}(D_\psi)$ (see for instance [7, Exercise 1, p. 296]) the same holds regarding f. To simplify notation, always denote by $\partial \psi(x)$ the subdifferential of ψ at x in the sense of convex analysis, while

$$D_{\partial \psi} := \{x \in X : \partial \psi(x) \neq \emptyset\} .$$

Theorem 23.5 of [7] gives $\mathrm{int}(D_\psi) = \mathrm{int}(D_{\partial \psi})$. Moreover, by Theorems 23.5 and 23.3 in [7], $\partial \psi(x)$ is always convex and weak* closed. We say that $x \in D_\psi$ is a critical point of f when

$$\Phi^0(x; z - x) + \psi(z) - \psi(x) \geq 0 \quad \forall z \in X .$$

If $\psi \equiv 0$, it clearly signifies $0 \in \partial \Phi(x)$, namely x is a critical point of Φ according to [5, Definition 2.1]Chang. The symbol $K(f)$ indicates the set of all critical points for f. Given a real number c, we write

$$K_c(f) := K(f) \cap f^{-1}(c) \qquad \text{and} \quad f_c := \{x \in X : f(x) \leq c\} .$$

If $K_c(f) \neq \emptyset$ then $c \in \mathbb{R}$ is said to be a critical value of f.

Let S be a nonempty closed subset of X. The function f is said to fulfil the Cerami condition at the level c and around the set S provided

$(\mathrm{C})_{S,c}$ Every sequence $\{x_n\} \subseteq X$ such that $d(x_n, S) \to 0$, $f(x_n) \to c$, and

$$(1 + \|x_n\|) \left(\Phi^0(x_n; x - x_n) + \psi(x) - \psi(x_n) \right) \geq -\epsilon_n \|x - x_n\| \tag{2}$$

for every $n \in \mathbb{N}$ and $x \in X$, where $\epsilon_n \to 0^+$, possesses a convergent subsequence.

When $S = X$ we simply write $(\mathrm{C})_c$ in place of $(\mathrm{C})_{S,c}$. Moreover, if $(\mathrm{C})_c$ is verified at any level c, we write (C).

The following Cerami condition at level c for functions $\Phi\colon X \to \mathbb{R}$ which are locally Lipschitz continuous was introduced in [12].

$(\mathcal{C})_c$ Every sequence $\{x_n\} \subseteq X$ such that $\Phi(x_n) \to c$ and

$$(1 + \|x_n\|)m(x_n) \to 0 \tag{3}$$

possesses a convergent subsequence.

The result below discusses the relationship between $(\mathrm{C})_c$ and $(\mathcal{C})_c$.

Proposition 2.1 *If* $\psi = 0, (\mathrm{C})_c$ *reduces to* $(\mathcal{C})_c$.

Proof It is sufficient to show the equivalence between (2) and (3). Assume that (2) holds. Then, there exists $x_n^* \in \partial\Phi(x_n)$ such that $m(x_n) = \|x_n^*\|$ for every $n \in \mathbb{N}$. Hence,

$$(1 + \|x_n\|)\Phi^0(x_n; x - x_n) \ge (1 + \|x_n\|)\langle x_n^*, x - x_n\rangle \ge -(1 + \|x_n\|)\|x_n^*\|\|x - x_n\|,$$

that is

$$(1 + \|x_n\|)\Phi^0(x_n; x - x_n) \ge -\epsilon_n\|x - x_n\|$$

for every $n \in \mathbb{N}, x \in X$, where $\epsilon_n = (1 + \|x_n\|)m(x_n) \to 0$ and (1) is verified (with $\psi = 0$).

Conversely, we admit that (1) (with $\psi = 0$) is satisfied. Since $\chi_n(z) = \frac{1+\|x_n\|}{\epsilon_n}\Phi^0(x_n; z)$ is continuous, convex, $\chi_n(0) = 0$ and $\chi_n(z) \ge -\|z\|$ for every $n \in \mathbb{N}, z \in X$, by Lemma 1.3 of [22] there exists $x_n^* \in X^*$ with $\|x_n^*\| \le 1$ such that

$$\chi_n(z) \ge \langle x_n^*, z\rangle$$

for every $n \in \mathbb{N}, z \in X$. Thus, $w_n^* = \frac{\epsilon_n}{1+\|x_n\|}x_n^* \in \partial\Phi(x_n)$ and

$$(1 + \|x_n\|)m(x_n) \le (1 + \|x_n\|)\|w_n^*\| = \epsilon_n\|x_n^*\| \le \epsilon_n,$$

so that (2) is verified. □

Let S be a nonempty, closed subset of X. The function f is said to fulfill the (PS) condition at the level c and around the set S is:

$(\mathrm{PS})_{S,c}$ Every sequence $\{x_n\} \subseteq X$ such that $d(x_n, S) \to 0$, $f(x_n) \to c$, and

$$\Phi^0(x_n; x - x_n) + \psi(x) - \psi(x_n) \ge -\epsilon_n\|x - x_n\| \tag{4}$$

for every $n \in \mathbb{N}, x \in X$, where $\epsilon_n \to 0^+$, possesses a convergent subsequence.

We want explicitly observe that condition $(\mathrm{C})_{S,c}$ is a weaker form of the $(\mathrm{PS})_{S,c}$ condition.

Proposition 2.2 *Let* (H_f) *and* $(\mathrm{PS})_{S,c}$ *be fulfilled. Then f satisfies condition* $(\mathrm{C})_{S,c}$.

Proof Let $\{x_n\} \subseteq X$ such that $d(x_n, S) \to 0, f(x_n) \to c$, and

$$(1 + \|x_n\|)\left(\Phi^0(x_n; x - x_n) + \psi(x) - \psi(x_n)\right) \ge -\epsilon_n\|x - x_n\|$$

for every $n \in \mathbb{N}$ and $x \in X$, where $\epsilon_n \to 0^+$. Hence,

$$\Phi^0(x_n; x - x_n) + \psi(x) - \psi(x_n) \geq -\bar{\epsilon}_n \|x - x_n\|$$

for every $n \in \mathbb{N}$ and $x \in X$, where $\bar{\epsilon}_n = \epsilon_n/(1 + \|x_n\|) \to 0^+$. At this point, by condition $(\mathrm{PS})_{S,c}$, $\{x_n\}$ admits a convergent subsequence and the proof is complete. □

Remark 2.1 It is simple to verify that condition $(\mathrm{C})_{S,c}$ is equivalent to $(\mathrm{PS})_{S,c}$ on bounded sets. In fact, if f satisfies condition $(\mathrm{C})_{S,c}$, $\{x_n\}$ is bounded and such that (4) holds, then there exists $M > 0$ with $\|x_n\| \leq M\ \forall n \in \mathbb{N}$, hence

$$\frac{1}{1+M}(\Phi^0(x_n; x - x_n) + \psi(x) - \psi(x_n)) \geq -\frac{\epsilon_n}{1+M}\|x - x_n\| \geq -\frac{\epsilon_n}{1 + \|x_n\|}\|x - x_n\|.$$

for every $n \in \mathbb{N}$ and $x \in X$. So that (2) holds, where $\epsilon_n(1 + M) \to 0^+$.

The following version of Theorem 2.2 of [13] will be particularly useful.

Theorem 2.1 *Let* (H_f) *be fulfilled, let* $\epsilon > 0$, *and let* B *and* C *be two nonempty closed sets in* X. *Suppose that* C *is compact,* $B \cap C = \emptyset$, $C \subseteq D_\psi$ *and fix* $M > \max\{1, \max_{x \in C}\|x\|\}$. *If, moreover,*

(a_1) *to each* $x \in C$ *there corresponds a point* $\xi_x \in X$ *such that*

$$(1 + \|x\|)(\Phi^0(x; \xi_x - x) + \psi(\xi_x) - \psi(x)) < -5\epsilon M\|\xi_x - x\|,$$

then for every $k > 1$ *there exist* $t_0 \in (0, 1]$, $\alpha \in C^0([0, 1] \times X, X)$ *and* $\varphi \in C^0(X, \mathbb{R}_0^+)$ *with the following properties:*

(i_1) $\alpha(t, D_\psi) \subseteq D_\psi \forall t \in [0, t_0)$ *and* $\alpha(t, x) = x\ \forall (t, x) \in [0, t_0) \times B$.
(i_2) $\|\alpha(t, x) - x\| \leq kt\ \forall (t, x) \in [0, t_0) \times X$.
(i_3) $f(\alpha(t, x)) - f(x) \leq -\frac{5\epsilon M}{1+M}\varphi(x)t\ \forall (t, x) \in [0, t_0) \times D_\psi$.
(i_4) $\varphi(x) = 1\ \forall x \in C$.

Proof Put $\sigma = \frac{5\epsilon M}{1+M}$ and verify that

(a_1') to each $x \in C$ there corresponds a point $\xi_x \in X$ such that

$$\Phi^0(x; \xi_x - x) + \psi(\xi_x) - \psi(x) < -\sigma\|\xi_x - x\|.$$

If $x \in C$ then, by (a_1), we can find a $\xi_x \in X$, with $\xi_x \neq x$, satisfying

$$\Phi^0(x; \xi_x - x) + \psi(\xi_x) - \psi(x) < -\frac{5\epsilon M}{1 + \|x\|}\|\xi_x - x\|. \tag{5}$$

Hence, (a_1') follows from (5) once one observes that $-\frac{5\epsilon M}{1+\|x\|} < -\sigma$. Now, the conclusion is achieved by applying Theorem 2.2 of [13] with σ in place of ϵ. □

The following version [9, pp. 444, 456] of the famous variational principle of Ekeland will be very useful.

Theorem 2.2 *Let* (Z, d) *be a complete metric space, and let* $\Pi: Z \to \mathbb{R} \cup \{+\infty\}$ *be a proper, lower semicontinuous function bounded from below. Then to every* $\epsilon, \delta > 0$ *and* $\bar{z} \in Z$ *satisfying* $\Pi(\bar{z}) < \inf_{z \in Z}\Pi(z) + \epsilon$, *there corresponds a point* $z_0 \in Z$ *such that*

$$\Pi(z_0) \leq \Pi(\bar{z}), \qquad d(z_0, \bar{z}) \leq 1/\delta, \qquad \Pi(z) - \Pi(z_0) \geq -\epsilon d(z, z_0) \quad \forall z \in Z.$$

3 Existence of critical points

The main result of this section, Theorem 2.1 below, is a suitable version of Theorem 3.1 of [13]. Let $f: X \to \mathbb{R} \cup \{+\infty\}$ be fulfil the structural hypothesis

(H'_f) $f(x) := \Phi(x) + \psi(x)$ for all $x \in X$, where $\Phi: X \to \mathbb{R}$ is locally Lipschitz continuous while $\psi: X \to \mathbb{R} \cup \{+\infty\}$ is convex, proper and lower semicontinuos. Moreover, ψ is continuous on any nonempty compact set $A \subseteq X$ such that $\sup_{x\in A} \psi(x) < +\infty$.

Let B be a closed subset of X, and let $\mathcal{F}$ be a class of compact sets in X. We say that $\mathcal{F}$ is a homotopy-stable family with extended boundary B when for every $A \in \mathcal{F}$ and every $\eta \in C^0([0,1] \times X, X)$ such that $\eta(t,x) = x$ in $(\{0\} \times X) \cup ([0,1] \times B)$ one has $\eta(\{1\} \times A) \in \mathcal{F}$. The following assumptions will be posited in the sequel:

(a_2) *$\mathcal{F}$ is a homotopy-stable family with extended boundary B, the function f fulfills condition* (H'_f), *and*

$$c = \inf_{A\in\mathcal{F}} \sup_{x\in A} f(x) < +\infty.$$

(a_3) *There exists a closed subset F of X such that*

$$(A \cap F) \setminus B \neq \emptyset \quad \forall A \in \mathcal{F}, \tag{6}$$

while moreover,

$$\sup_{x\in B} f(x) \leq \inf_{x\in F} f(x). \tag{7}$$

Both (a_2) and (a_3) imply that

$$\inf_{x\in F} f(x) \leq c. \tag{8}$$

Theorem 3.1 *Let* (a_2) *and* (a_3) *be satisfied. Then to every sequence* $\{A_n\} \subseteq \mathcal{F}$ *such that* $\lim_{n\to\infty} \sup_{x\in A_n} f(x) = c$ *there corresponds a sequence* $\{x_n\} \subseteq X \setminus B$ *having the following properties:*

(i_5) $\lim_{n\to\infty} f(x_n) = c$.
(i_6) $(1 + \|x_n\|)(\Phi^0(x_n; x - x_n) + \psi(x) - \psi(x_n)) \geq -\epsilon_n \|x - x_n\|,\ \forall n \in \mathbb{N}, x \in X,$ *where* $\epsilon_n \to 0^+$.
(i_7) $\lim_{n\to\infty} d(x_n, F) = c$ *provided* $\inf_{x\in F} f(x) = c$.
(i_8) $\lim_{n\to\infty} d(x_n, A_n) = 0$.

Proof The reasoning is chiefly adapted from that in [10, 13] to establish Theorems 1 and 3.1, respectively, but, for the reader convenience, we report here some details. We begin by considering the case

$$\inf_{x\in F} f(x) = c. \tag{9}$$

Pick an $\epsilon > 0$ and choose $A_\epsilon \in \mathcal{F}$ such that

$$c \leq \sup_{x\in A_\epsilon} f(x) < c + \frac{\epsilon^2}{8}. \tag{10}$$

We shall look for a point $x_\epsilon \in X \setminus B$ such that

$$c - \frac{\epsilon^2}{8} \le f(x_\epsilon) < c + \frac{5}{4}\epsilon^2, \tag{11}$$

$$(1 + \|x_n\|)(\Phi^0(x_\epsilon; x - x_\epsilon) + \psi(x) - \psi(x_\epsilon)) \ge -5\epsilon\chi(\epsilon)\|x - x_\epsilon\| \quad \forall x \in X, \tag{12}$$

$$d(x_\epsilon, F) \le \frac{3}{2}\epsilon, \tag{13}$$

$$d(x_\epsilon, A_\epsilon) \le \frac{\epsilon}{2}, \tag{14}$$

where $0 < \chi(\epsilon) < 1$, which obviously provides a sequence $\{x_n\} \subseteq X \setminus B$ enjoying properties $(i_5) - (i_8)$. Put

$$A'_\epsilon = \{1\} \times A_\epsilon, \quad F_\epsilon = N_\epsilon(F), \quad G_\epsilon = (\{0\} \times X) \cup ([0,1] \times ((A_\epsilon \setminus F_\epsilon) \cup B))$$

and denote by $\mathcal{L}$ the space of all $\eta \in C^0([0,1] \times X, X)$ such that

$$\eta(t,x) = x \ \ \forall (t,x) \in G_\epsilon, \qquad \sup_{(t,x)\in[0,1]\times X} \|\eta(t,x) - x\| < +\infty.$$

Clearly, $(\{0\} \times X) \cup ([0,1] \times B) \subseteq G_\epsilon$. Hence,

$$\eta(A'_\epsilon) \in \mathcal{F} \ \ \forall \eta \in \mathcal{L}. \tag{15}$$

It is a simple matter to verify that $\mathcal{L}$, equipped with the metric ρ of the uniform convergence, is complete. Finally, for every $x \in X$, let us consider

$$f_1(x) = \max\{0, \epsilon^2 - \epsilon d(x, F)\}, \ \ f_2(x) = \min\left\{\frac{\epsilon^2}{8}, \epsilon d(x, ((A_\epsilon \setminus F_\epsilon) \cup B)\right\}$$

$$g(x) = f(x) + f_1(x) + f_2(x)$$

and define

$$I(\eta) = \sup_{x\in\eta(A'_\epsilon)} g(x), \ \ \forall \eta \in \mathcal{L}.$$

The function $I \colon \mathcal{L} \to \mathrm{I\!R} \cup \{+\infty\}$ is lower semicontinuos. Let $\overline{\eta}(t,x) = x$ for every $(t,x) \in [0,1] \times X$. Gathering together (15), (6) and (9) one has

$$c + \epsilon^2 \le \inf_{\eta\in\mathcal{L}} I(\eta) \tag{16}$$

as well as

$$I(\overline{\eta}) < \inf_{\eta\in\mathcal{L}} I(\eta) + \frac{\epsilon^2}{4}. \tag{17}$$

By Theorem 2.2, there exists $\eta_0 \in \mathcal{L}$ such that

$$I(\eta_0) \le I(\overline{\eta}), \tag{18}$$

$$\rho(\eta_0, \overline{\eta}) \le \frac{\epsilon}{2}, \tag{19}$$

$$I(\eta) \geq I(\eta_0) - \frac{\epsilon}{2}\rho(\eta, \eta_0), \quad \forall \eta \in \mathcal{L}. \tag{20}$$

From (18) it follows that

$$\sup_{x \in \eta_0(A'_\epsilon)} \psi(x) \leq I(\overline{\eta}) - \min_{x \in \eta_0(A'_\epsilon)} \Phi(x) < +\infty.$$

Hence, in view of (H'_f), the function $x \mapsto \psi(\eta_0(1,x))$ is continuous on A_ϵ, as well as $x \mapsto g(\eta_0(1,x))$ $\forall x \in A_\epsilon$. At this point, we can consider the following nonempty compact set

$$C_\epsilon = \left\{ w \in \eta_0(A'_\epsilon) : g(w) = \max_{x \in \eta_0(A'_\epsilon)} g(x) \right\}.$$

Let us show that there exists $z_0 \in (\eta_0(A'_\epsilon \cap F))\backslash B$ in such a way that

$$f(z_0) = \max_{x \in \eta_0(A'_\epsilon) \cap F} f(x). \tag{21}$$

Let $\widehat{z} \in \eta_0(A'_\epsilon) \cap F$ be with $f(\widehat{z}) = \max_{x \in \eta_0(A'_\epsilon) \cap F} f(x)$. If $\widehat{z} \notin B$, (21) is true with $z_0 = \widehat{z}$. Otherwise, owing to (6), there exists $z_0 \in (\eta_0(A'_\epsilon) \cap F)\backslash B$. Hence, by (7),

$$\max_{x \in \eta_0(A'_\epsilon) \cap F} f(x) = f(\widehat{z}) \leq \sup_{x \in B} f(x) \leq \inf_{x \in F} f(x) \leq f(z_0) \leq \max_{x \in \eta_0(A'_\epsilon) \cap F} f(x)$$

and (21) is proved. Define $B' = (A_\epsilon \backslash F_\epsilon) \cap B$. Using (21) and reasoning as in [10, pp. 445–446] we obtain that

$$B' \cap C_\epsilon = \emptyset.$$

Fix $M_\epsilon > \max\{1, \max_{w \in C_\epsilon} \|w\|\}$. We claim that there exists $x_\epsilon \in C_\epsilon$ satisfying

$$(1 + \|x_\epsilon\|)(\Phi^0(x_\epsilon; x - x_\epsilon) + \psi(x) - \psi(x_\epsilon)) \geq -\frac{5\epsilon M_\epsilon}{1 + M_\epsilon}\|x - x_\epsilon\| \quad \forall x \in X. \tag{22}$$

Suppose (22) is false. Observe that $1 < \frac{2M_\epsilon}{1+M_\epsilon}$ and fix $k \in]1, \frac{2M_\epsilon}{1+M_\epsilon}[$. Applying Theorem 2.1 to the sets B' and C, $(\mathrm{i}_1) - (\mathrm{i}_4)$ hold true for suitable $t_0 \in]0,1]$, $\alpha \in C^0([0,1] \times X, X)$ and $\varphi \in C^0(X, \mathbb{R}_0^+)$. Pick $\lambda \in [0, t_1]$, where $0 < t_1 < t_0$, and define

$$\eta_\lambda(t,x) = \alpha(\lambda t, \eta_0(t,x)), \quad (t,x) \in [0,1] \times X.$$

Making use of (i_1) and (i_2), it is easy to see that $\eta_\lambda \in \mathcal{L}$ and $\varphi(\eta_0, \eta_\lambda) \leq \lambda k$. Hence, from (20) one has

$$I(\eta_\lambda) \geq I(\eta_0) - \lambda k \frac{\epsilon}{2}. \tag{23}$$

Moreover, in view of (i_3), it follows that

$$\sup_{x \in \eta_\lambda(A'_\epsilon)} f(x) \leq I(\eta_0) - \frac{5\epsilon M_\epsilon}{1 + M_\epsilon}\lambda \min_{x \in \eta_0(A'_\epsilon)} \varphi(x) < +\infty.$$

So, the functions $x \mapsto \psi(\eta_\lambda(1,x))$ and $x \mapsto g(\eta_\lambda(1,x)), x \in A_\epsilon$ are continuous and there exists $x_\lambda \in A_\epsilon$ such that $g(\eta_\lambda(1, x_\lambda)) = I(\eta_\lambda)$. Thanks to (23) we get

$$g(\eta_\lambda(1, x_\lambda)) - g(\eta_0(1,x)) \geq -\lambda k \frac{\epsilon}{2} \quad \forall x \in A_\epsilon \tag{24}$$

and (i_3), (24) provides

$$-\lambda k\frac{\epsilon}{2} \leq g(\eta_\lambda(1,x_\lambda)) - g(\eta_0(1,x_\lambda)) \leq -\frac{5\epsilon M_\epsilon}{1+M_\epsilon}\lambda\varphi(\eta_0(1,x_\lambda)) + 2\epsilon\lambda k,$$

that is

$$\varphi(\eta_0(1,x_\lambda)) \leq \frac{M_\epsilon+1}{2M_\epsilon}k \quad \forall\lambda \in [0,t_1]. \tag{25}$$

Let $\widehat{x} \in A_\epsilon$ a cluster point of $\{x_\lambda : \lambda \in [0,t_1]\}$. Since $(\lambda,x) \mapsto g(\eta_\lambda(1,x))$ is continuous on $[0,t_1] \times \overline{\{x_\lambda : \lambda \in [0,t_1]\}}$, letting $\lambda \to 0^+$ in (24) we obtain

$$g(\eta_0(1,\widehat{x})) - g(\eta_0(1,x)) \geq 0, \quad \forall x \in A_\epsilon$$

namely $\widehat{x} \in C_\epsilon$. This implies $\varphi(\eta_0(1,\widehat{x})) = 1$, against (25) which forces $\varphi(\eta_0(1,\widehat{x})) \leq \frac{1+M_\epsilon}{2M_\epsilon}k < 1$. Hence (22) is true. Let $x_\epsilon \in C_\epsilon$ satisfy (22). Then $x_\epsilon \notin B$ and (12) is verified with $\chi(\epsilon) = \frac{M\epsilon}{1+M\epsilon}$. Obviously, $x_\epsilon = \eta_0(1,\overline{x})$ with $\overline{x} \in A_\epsilon$ and $\overline{x} \in F_\epsilon$. In fact, if $\overline{x} \in A_\epsilon \setminus F_\epsilon \subseteq B'$ then $x_\epsilon = \overline{x} \in C \cap B'$, which is absurd. At this point, by (19)

$$d(x_\epsilon, F) \leq \|\eta_0(1,\overline{x}) - \overline{\eta}(1,\overline{x})\| + \epsilon \leq \frac{3}{2}\epsilon$$

$$d(x_\epsilon, A_\epsilon) \leq \|\eta_0(1,\overline{x}) - \overline{\eta}(1,\overline{x})\| \leq \frac{\epsilon}{2}$$

and (13), (14) are satisfied. It remains to verify (11). In order to do this, taking in mind the choice of x_ϵ, (18), (10) and the properties of f_1 and f_2, we obtain

$$f(x_\epsilon) \leq g(x_\epsilon) = I(\eta_0) \leq I(\overline{\eta}) = \sup_{x\in A_\epsilon} g(x) < c + \frac{\epsilon^2}{8} + \epsilon^2 + \frac{\epsilon^2}{8} = c + \frac{5}{2}\epsilon^2.$$

On the other hand, exploiting (16) yields

$$f(x_\epsilon) = I(\eta_0) - f_1(x_\epsilon) - f_2(x_\epsilon) \geq \inf_{\eta\in\mathcal{L}} I(\eta) - \epsilon^2 - \frac{\epsilon^2}{8} \geq c - \frac{\epsilon^2}{8}$$

and the proof is complete under condition (9).
Assume now that

$$\inf_{x\in F} f(x) < c. \tag{26}$$

Pick an $\epsilon > 0$ and choose $A_\epsilon \in \mathcal{F}$ such that

$$c \leq \sup_{x\in A_\epsilon} f(x) < c + \frac{\epsilon^2}{4}. \tag{27}$$

We shall look for a point $x_\epsilon \in X \setminus B$ such that

$$c \leq f(x_\epsilon) < c + \frac{\epsilon^2}{4}, \tag{28}$$

$$(1 + \|x_\epsilon\|)(\Phi^0(x_\epsilon; x - x_\epsilon) + \psi(x) - \psi(x_\epsilon)) \geq -5\epsilon\chi(\epsilon)\|x - x_\epsilon\| \ \forall x \in X. \tag{29}$$

$$d(x_\epsilon, A_\epsilon) \le \frac{\epsilon}{2} \tag{30}$$

where $0 < \chi(\epsilon) < 1$, which obviously provides a sequence $\{x_n\} \subseteq X \setminus B$ enjoying properties (i_5)–(i_8).
Denote by $\mathcal{L}$ the space of all $\eta \in C^0([0,1] \times X, X)$ such that

$$\eta(t,x) = x \quad \forall (t,x) \in (\{0\} \times X) \cup ([0,1] \times B), \qquad \sup_{(t,x)\in[0,1]\times X} \|\eta(t,x) - x\| < +\infty.$$

It is obvious that $\eta(A'_\epsilon) \in \mathcal{F}\ \forall \eta \in \mathcal{L}$ as well as $\mathcal{L}$, equipped with the metric ρ of the uniform convergence, is complete. The function $I : \mathcal{L} \to \mathrm{I\!R} \cup \{+\infty\}$ defined by putting

$$I(\eta) = \sup_{x\in\eta(A'_\epsilon)} f(x) \quad \forall \eta \in \mathcal{L}$$

is lower semicontinuous, bounded from below by c and, thanks to (27), satisfies (17). By Theorem 2.2, there exists $\eta_0 \in \mathcal{L}$ such that (18)–(20) hold. Hence,

$$\sup_{x\in\eta_0(A'_\epsilon)} f(x) = I(\eta_0) \le I(\overline{\eta}) < +\infty$$

and the function $x \mapsto f(\eta_0(1,x))$ is continuous on A_ϵ. Then, the set

$$C_\epsilon = \{w \in \eta_0(A'_\epsilon) : f(w) = \max_{x\in\eta_0(A'_\epsilon)} f(x)\}$$

is nonempty and compact. Putting together (7) and (26) we obtain

$$\sup_{x\in B} f(x) \le \inf_{x\in F} f(x) < c \le \inf_{x\in C_\epsilon} f(x),$$

which implies $B \cap C_\epsilon = \emptyset$. The same reasoning as before gives a point $x_\epsilon \in C_\epsilon$ such that

$$(1 + \|x_\epsilon\|)(\Phi^0(x_\epsilon; x - x_\epsilon) + \psi(x) - \psi(x_\epsilon)) \ge -5\epsilon\chi(\epsilon)\|x - x_\epsilon\| \quad \forall x \in X,$$

where $\chi(\epsilon) = \frac{M_\epsilon}{1+M_\epsilon}$, with $M_\epsilon > \{1, \max_{x\in C_\epsilon} \|x\|\}$, and (29) is proved. Moreover, $x_\epsilon \in C_\epsilon$ and by (18) and (27) one has

$$c \le \inf_{x\in C_\epsilon} f(x) \le f(x_\epsilon) = I(\eta_0) \le I(\overline{\eta}) = \sup_{x\in A_\epsilon} f(x) < c + \frac{\epsilon^2}{4}.$$

Finally, (30) can be achieved as in the preceding case. □

A meaningful consequence of Theorem 2.1 is the following

Theorem 3.2 *Let* (a_2) *and* (a_3) *be satisfied. Suppose that either* $(C)_c$ *or* $(C)_{F,c}$ *holds according to whether* $\inf_{x\in F} f(x) < c$ *or* $\inf_{x\in F} f(x) = c$*. Then* $K_c(f) \ne \emptyset$*. If moreover,* $\inf_{x\in F} f(x) = c$*, then* $K_c(f) \cap F \ne \emptyset$*.*

Proof Theorem 3.1 provides a sequence $\{x_n\} \subseteq X \setminus B$ with properties (i_5)–(i_8). On account of the Cerami condition we may assume that $x_n \to x$ in X, where a subsequence is considered when necessary. The conclusion follows from (i_5) to (i_7). □

Suppose Q denotes a compact set in X, Q_0 is a nonempty closed subset of $Q, \gamma_0 \in C^0(Q_0, X), \Gamma = \{\gamma \in C^0(Q, X) : \gamma|_{Q_0} = \gamma_0\}, \mathcal{F} = \{\gamma(Q) : \gamma \in \Gamma\}$, and $B = \gamma_0(Q_0)$. From Theorem 3.2 we obtain the following

Theorem 3.3 *Let the function* $f : X \to \mathbb{R} \cup \{+\infty\}$ *fulfill the following assumptions in addition to* (H'_f).

(a_4) $\sup_{x\in Q} f(\gamma(x)) < +\infty$ *for some* $\gamma \in \Gamma$.
(a_5) *There exists a closed subset F of X such that* $(\gamma(Q) \cap F)\setminus\gamma_0(Q_0) \neq \emptyset\ \forall\gamma \in \Gamma$ *and, moreover,* $\sup_{x\in Q_0} f(\gamma_0(x)) \leq \inf_{x\in F} f(x)$.
(a_6) *Setting* $c = \inf_{\gamma\in\Gamma} \sup_{x\in Q} f(\gamma(x))$, *either* $(C)_c$ *or* $(C)_{F,c}$ *is satisfied, according to whether* $\inf_{x\in F} f(x) < c$ *or* $\inf_{x\in F} f(x) = c$.

Then the conclusion of Theorem 3.2 holds true.

Remark 3.1 Making use of Theorems 3.2 and 3.3 all the structure results contained in [13] can be stated with $(\mathrm{PS})_c$ replaced by $(\mathrm{C})_c$.

4 An Application

Let Ω be a nonempty, bounded, open subset of the Euclidean N-space $(\mathbb{R}^N, |\cdot|)$, $N \geq 3$, having a smooth boundary $\partial\Omega$. Let $H^1_0(\Omega)$ be the closure of $C^\infty(\Omega)$ in $W^{1,2}(\Omega)$ and let us consider the norm

$$\|u\| := \left(\int_\Omega |\nabla u(x)|^2 dx\right)^{1/2} \quad \forall u \in H^1_0(\Omega).$$

Denote by 2^* the critical exponent for the Sobolev embedding $H^1_0(\Omega) \hookrightarrow L^p(\Omega)$. Recall that $2^* = \frac{2N}{N-2}$, if $p \in [1, 2^*]$ then there exists a positive constant c_p such that

$$\|u\|_p \leq c_p\|u\|, \quad u \in H^1_0(\Omega) \tag{31}$$

and in particular, the embedding is compact whenever $p \in [1, 2^*[$ (see, e.g. [20, Proposition B.7]). Consider the eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u \quad \text{on} \quad \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{32}$$

It is known that (32) possesses a sequence $\{\lambda_n\}$ of positive eigenvalues fulfilling $\lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$, and let $\{\varphi_n\}$ be a corresponding sequence of eigenfunctions normalized as follows

$$\|\varphi_n\|^2 = 1 = \lambda_n\|\varphi_n\|_2^2, \quad n \in \mathbb{N};$$
$$\int_\Omega \nabla\varphi_m(x)\cdot\nabla\varphi_n(x)\mathrm{d}x = \int_\Omega \varphi_m(x)\varphi_n(x)\mathrm{d}x = 0 \,\text{provided}\, m \neq n. \tag{33}$$

Assume that $g : \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:

(g_1) g is measurable.
(g_2) There exists $a > 0, p \in]2, 2^*]$ such that $|g(t)| \leq a(1 + |t|^{p-1})$ for every $t \in \mathbb{R}$.

Then the functions $G : \mathbb{R} \to \mathbb{R}$ and $\mathcal{G} : H^1_0(\Omega) \to \mathbb{R}$ given by

$$G(\xi) := \int_0^\xi g(t)\mathrm{d}t \quad \forall\xi \in \mathbb{R}, \quad \mathcal{G}(u) := \int_\Omega G(u(x))\mathrm{d}x \quad \forall u \in H^1_0(\Omega)$$

turn out well defined and locally Lipschitz continuous. For this reason, it makes sense to consider their generalized directional derivatives G^0 and $\mathcal{G}^0$. Moreover, on account of formula (9) at p. 84 in [6], one has

$$\mathcal{G}^0(u;v) \le \int_\Omega G^0(u(x);v(x))\mathrm{d}x, \quad u,v \in H_0^1(\Omega). \tag{34}$$

Thanks to (g_2), exploiting Theorem 14 of [17], leads to

$$|G(\xi)| \le 2a(1+|\xi|^p) \quad \text{for any } \xi \in \mathbb{R}. \tag{35}$$

Let $s \ge 2$ be an integer such that $\lambda_{s-1} < \lambda_s$ and let

$$X_2 := \operatorname{span}\{\varphi_1,\dots,\varphi_s\},\ X_1 := X_2^\perp. \tag{36}$$

Given a closed and convex subset K of $H_0^1(\Omega)$ such that

$$X_2 \subseteq K, \tag{37}$$

consider the following elliptic variational-hemivariational inequality problem:

(P_{λ_s}) Find $u \in K$ such that

$$-\int_\Omega \nabla u(x)\cdot\nabla(v-u)(x)\mathrm{d}x + \lambda_s\int_\Omega u(x)(v-u)(x)\mathrm{d}x \le \mathcal{G}^0(u;v-u)$$

for all $v \in K$.

In order to solve problem (P_{λ_s}) we will further assume that

(g_2') Hypothesis (g_2) holds with $a \le \frac{\alpha}{8(m(\Omega)+c_p^p)}$, where $\alpha = \lambda_s/\lambda_{s-1} - 1$.
(g_3) $\limsup_{|\xi|\to+\infty} \frac{G(\xi)}{|\xi|^2} < +\infty$.
(g_4) $\lim_{|\xi|\to+\infty} \sup_{y\in\partial G(\xi)}(2G(\xi) - y\xi) = -\infty$,
(g_5) $G(\xi) \ge \int_0^1 g(t)\mathrm{d}t \quad \forall\xi \in \mathbb{R}$.

Due to (34), any solution u of (P_{λ_s}) also fulfils the inequality

$$-\int_\Omega \nabla u\cdot\nabla(v-u)\mathrm{d}x + \lambda_s\int_\Omega u(v-u)\mathrm{d}x \le \int_\Omega G^0(u(x);(v-u)(x))\mathrm{d}x,$$

for all $v \in K$.
In particular, when g is continuous, while $K := H_0^1(\Omega)$, the function $u \in H_0^1(\Omega)$ turns out a weak solution to the Dirichlet problem

$$-\Delta u - \lambda_s u + g(u) = 0 \quad \text{in} \quad \Omega, \qquad u|_{\partial\Omega} = 0,$$

which has been previously investigated in [3].

Theorem 4.1 *Suppose* $(g_1), (g_2'), (g_3) - (g_5)$ *hold true. Then problem* (P_{λ_s}) *admits at least one solution.*

Proof Let us define

$$\overline{G}(\xi) := G(\xi) - \int_0^1 g(t)\mathrm{d}t \quad \forall\xi \in \mathbb{R}, \qquad \overline{\mathcal{G}}(u) := \int_\Omega \overline{G}(u(x))\mathrm{d}x \quad \forall u \in H_0^1(\Omega).$$

Let $X_1,\ X_2$ be as in (36). Put $X := X_1 \oplus X_2 = H^1_0(\Omega)$ and, for every $u \in X$, define

$$\Phi(u) := \frac{1}{2}\int_\Omega |\nabla u(x)|^2 \mathrm{d}x - \frac{\lambda_s}{2}\int_\Omega u(x)^2 \mathrm{d}x + \overline{\mathcal{G}}(u),$$

as well as

$$\psi(u) := \begin{cases} 0, & \text{if } u \in K, \\ +\infty & \text{otherwise,} \end{cases} \qquad f(u) := \Phi(u) + \psi(u),$$

where K is as in (37). It is easy to verify that Φ is locally Lipschitz continuous. Hence, f satisfies condition (H'_f). We shall prove that f fulfils condition (C). Pick a sequence $\{u_n\} \subset X$ such that $\{f(u_n)\}$ converges to some $\bar{c} \in \mathbb{R}$ and

$$(1 + \|u_n\|)\left(\Phi^0(u_n; v - u_n) + \psi(v) - \psi(u_n)\right) \geq -\epsilon_n \|v - u_n\| \tag{38}$$

for every $n \in \mathbb{N}$ and $v \in X$, where $\epsilon_n \to 0^+$.

We claim that $\{u_n\}$ is bounded. If the assertion were false, we could suppose that

$$\lim_{n \to +\infty} \|u_n\| = +\infty, \tag{39}$$

where a subsequence is considered if necessary. Define $w_n := \frac{u_n}{\|u_n\|}$ for every $n \in \mathbb{N}$. Since $\|w_n\| = 1$, we may suppose that

$$w_n \rightharpoonup w \quad \text{in } X, \qquad w_n \to w \quad \text{in } L^2(\Omega) \quad \text{and a.e.}. \tag{40}$$

We shall prove that there exists $\rho > 0$ such that

$$\frac{\overline{G}(u_n(x))}{\|u_n\|^2} \leq 4a\left(\frac{1}{\|u_n\|^2} + \rho^{p-2}|w_n(x)|^2\right) \qquad \forall n \in \mathbb{N},\ x \in \Omega. \tag{41}$$

Indeed, by (g_3) there exist a positive number δ such that, for a suitable $\rho > 0$,

$$\sup_{|\xi| > \rho} \frac{\overline{G}(\xi)}{|\xi|^2} < \delta.$$

Hence,

$$\frac{\overline{G}(u_n(x))}{\|u_n\|^2} < \delta |w_n(x)|^2, \tag{42}$$

whenever $|u_n(x)| > \rho$. It is not restrictive to suppose that $\rho > (\delta/(4a))^{\frac{1}{p-2}}$. At this point, (g_2) and condition (35) ensures that

$$|\overline{G}(\xi)| \leq 4a(1 + |\xi|^p) \quad \text{for any } \xi \in \mathbb{R}, \tag{43}$$

hence

$$\frac{\overline{G}(u_n(x))}{\|u_n\|^2} \leq 4a\left(\frac{1}{\|u_n\|^2} + |u_n(x)|^{p-2}|w_n(x)|^2\right) \tag{44}$$

for every $n \in \mathbb{N}$ and $x \in \Omega$. If $|u_n(x)| > \rho$, condition (42) forces

$$\frac{\overline{G}(u_n(x))}{\|u_n\|^2} < 4a\rho^{p-2}|w_n(x)|^2 < 4a\left(\frac{1}{\|u_n\|^2} + \rho^{p-2}|w_n(x)|^2\right).$$

Otherwise, (41) is an immediate consequence of (44).

Let $\Omega_0 := \{x \in \Omega : \ w(x) = 0\}$. Obviously, thanks to (39) one has

$$|u_n(x)| \to +\infty \quad \forall x \in \Omega \setminus \Omega_0. \tag{45}$$

Let us verify that

$$m(\Omega \setminus \Omega_0) > 0. \tag{46}$$

If $m(\Omega \setminus \Omega_0) = 0$ then, since (g_5) implies that

$$\overline{G}(\xi) \geq 0 \quad \forall \xi \in \mathbb{R}, \tag{47}$$

putting together (39)–(41), provides $\frac{\overline{G}(u_n(\cdot))}{\|u_n\|^2} \to 0$ a.e. in Ω and

$$\frac{\overline{G}(u_n(\cdot))}{\|u_n\|^2} \to 0 \quad \text{in } L^1(\Omega). \tag{48}$$

Observe next, since $\{f(u_n)\}$ is bounded, there exists a positive number M such that

$$\Phi(u_n) = f(u_n) \leq M \quad \forall n \in \mathbb{N},$$

that is

$$\frac{1}{2}\int_\Omega |\nabla u_n(x)|^2 \mathrm{d}x - \frac{\lambda_s}{2}\int_\Omega u_n^2(x)\mathrm{d}x + \int_\Omega \overline{G}(u_n(u))\mathrm{d}x \leq M \quad \forall n \in \mathbb{N}.$$

Hence,

$$\frac{1}{2} - \frac{\lambda_s}{2}\|w_n\|_2^2 + \int_\Omega \frac{\overline{G}(u_n(x))}{\|u_n\|^2} dx \leq \frac{M}{\|u_n\|^2} \quad \forall n \in \mathbb{N}. \tag{49}$$

At this point, bearing in mind (39), (48), (40) and the fact that $m(\Omega \setminus \Omega_0) = 0$, condition (49) leads to a contradiction, and (46) is proved.

It is obvious that inequality (38) can be equivalently written as

$$(1 + \|u_n\|)\, \Phi^0(u_n; v - u_n) \geq -\epsilon_n \|v - u_n\| \tag{50}$$

for every $n \in \mathbb{N}$ and $v \in K$, where $\epsilon_n \to 0^+$. Exploiting (50) with $v = 0$, condition (34), as well as $\{u_n\} \subseteq K$ one has

$$\begin{aligned}
-\epsilon_n \frac{\|u_n\|}{1 + \|u_n\|} &\leq -\|u_n\|^2 + \lambda_s \|u_n\|_2^2 + \overline{\mathcal{G}}^0(u_n; -u_n) \\
&\leq -2\Phi(u_n) + 2\int_\Omega \overline{G}(u_n(x))\mathrm{d}x + \int_\Omega \overline{G}^0(u_n(x); -u_n(x))\mathrm{d}x \\
&= -2f(u_n) + \int_\Omega \Big(2\overline{G}(u_n(x)) - \langle z_n(x), u_n(x)\rangle\Big)\, \mathrm{d}x
\end{aligned} \tag{51}$$

for every $n \in \mathbb{N}$, where $z_n(x) \in \partial\overline{G}(u_n(x))$. Gathering ($g_2$) and ($g_4$) together yields a constant $\bar{M} > 0$ such that

$$\sup_{y \in \partial\overline{G}(\xi)} (2\overline{G}(\xi) - y\xi) \leq \bar{M} \quad \forall \xi \in \mathbb{R}. \tag{52}$$

Through (51) and (52) we obtain

$$-\epsilon_n \frac{\|u_n\|}{1 + \|u_n\|} \leq -2f(u_n) + m(\Omega_0)\bar{M} + \int_{\Omega \setminus \Omega_0} \sup_{y \in \partial\overline{G}(u_n(x))} (2\overline{G}(u_n(x)) - yu_n(x))\mathrm{d}x$$

for every $n \in \mathbb{N}$. Hence, from (45), (g_4), (46) and the above inequality it follows

$$f(u_n) \to -\infty,$$

against the choice of $\{u_n\}$ which, at this point, is proved to be bounded. Thus, passing to a subsequence if necessary, we may suppose both $u_n \rightharpoonup u$ in X and $u_n \to u$ in $L^2(\Omega)$. Since K is convex and closed $u \in K$ and exploiting (50) with $v = u$ one has

$$\begin{aligned} -\epsilon_n \frac{\|u_n\|}{1+\|u_n\|} + \|u_n\|^2 \leq & \int_\Omega \nabla u_n(x) \cdot \nabla u(x) \mathrm{d}x - \lambda_s \int_\Omega u_n(x)(u(x) - u_n(x)) \mathrm{d}x \\ & + \overline{\mathcal{G}}^0(u_n; u - u_n) \end{aligned} \tag{53}$$

for every $n \in \mathbb{N}$. Thanks to the upper semicontinuity of $\overline{\mathcal{G}}^0$ on $L^2(\Omega) \times L^2(\Omega)$ and the convergence of $\{u_n\}$ inequality (53) yields

$$\limsup_{n \to +\infty} \|u_n\|^2 \leq \|u\|^2$$

that is $u_n \to u$ in X and condition (C) is satisfied.

Put

$$Q := \{u \in X : \ \|u\| \leq 1\} \cap X_2, \quad Q_0 = \partial Q, \quad F := X_1,$$

$$\gamma_0 = id_{|Q_0}, \quad \Gamma := \{\gamma \in C^0(Q, X) : \ \gamma_{|Q_0} = \gamma_0\}.$$

Fix $u \in Q$ and observe that, making use of (33), (43) and (31) one has

$$\begin{aligned} f(u) = \Phi(u) &= \frac{1}{2}(\|u\|^2 - \lambda_s \|u\|_2^2) + \int_\Omega \overline{G}(u(x)) \mathrm{d}x \\ &\leq \frac{1}{2} \sum_{i=1}^{s} \left(1 - \frac{\lambda_s}{\lambda_i}\right) t_i^2 + 4a \int_\Omega (1 + |u(x)|^p) \mathrm{d}x \\ &\leq 4a(m(\Omega) + c_p^p). \end{aligned} \tag{54}$$

Hence

$$\sup_{u \in Q} f(u) < +\infty.$$

Thanks to Proposition 2.1 of [3] we get

$$(\gamma(Q) \cap F) \setminus \gamma_0(Q_0) \neq \emptyset \quad \forall \gamma \in \Gamma.$$

Let now $u \in F$. From (47) and (33) again one has

$$f(u) \geq \Phi(u) = \frac{1}{2} \sum_{i=s+1}^{+\infty} \left(1 - \frac{\lambda_s}{\lambda_i}\right) t_i^2 + \int_\Omega \overline{G}(u(x)) \mathrm{d}x \geq 0.$$

Thus,

$$\inf_{u \in F} f(u) \geq 0.$$

Finally, fix $u \in Q_0$, that is $u \in X_2$ and $\|u\| = 1$. Reasoning as in (54) and exploiting (g_2') we obtain

$$\begin{aligned} f(u) &\leq \frac{1}{2}\sum_{i=1}^{s}\left(1-\frac{\lambda_s}{\lambda_i}\right)t_i^2 + 4a(m(\Omega)+\|u\|_p^p) \\ &\leq -\frac{1}{2}\alpha\|u\|^2 + 4a(m(\Omega)+c_p^p\|u\|^p) \\ &= -\frac{1}{2}\alpha + 4a(m(\Omega)+c_p^p) \leq 0. \end{aligned}$$

Hence,

$$\sup_{u\in Q_0} f(u) \leq 0 \leq \inf_{u\in F} f(u).$$

At this point, Theorem 3.3 can be applied. So, there exists at least a point $u \in X$ such that

$$\Phi^0(u; v-u) + \psi(v) - \psi(u) \geq 0$$

for all $v \in X$. By definition of ψ, it follows that $u \in K$ and $\Phi^0(u; v-u) \geq 0$ for all $v \in K$, namely u is a solution to problem (P_{λ_s}) once observed that $\mathcal{G}^0 = \overline{\mathcal{G}}^0$, and the proof is complete. □

Example 4.1 Let $s \geq 2$ be an integer such that $\lambda_{s-1} < \lambda_s$ and $p \in]2, 2^*[$. Fix a positive number θ such that $\theta \leq \frac{\alpha}{8(m(\Omega)+c_p^p)}$, where $\alpha = \lambda_s/\lambda_{s-1} - 1$. Define the function $g : \mathbb{R} \to \mathbb{R}$ by setting

$$g(t) := \begin{cases} \theta e^2\left(t - \frac{|t|}{t}\right)e^{-|t|}, & |t| \leq 2, \quad t \neq 0 \\ \theta, & t = 0, \\ \theta t - \theta\frac{|t|}{t}, & |t| > 2. \end{cases}$$

A simple computation shows that, in this case one has

$$G(\xi) = \begin{cases} -\theta e^2|\xi|e^{-|\xi|}, & |\xi| \leq 2, \\ \frac{\theta}{2}\xi^2 - \theta|\xi| - 2\theta, & |\xi| > 2 \end{cases}$$

and all the assumptions of Theorem 4.1 are satisfied. Hence, problem (P_{λ_s}) admits at least a nontrivial solution.

Acknowledgements The authors would like to warmly thank S.A. Marano for suggesting them the study of the subject. Work performed under the auspicies of G.N.A.M.P.A., of I.N.D.A.M. and partially supported by M.I.U.R. of Italy, 2005.

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